Measure Theory with Ergodic Horizons
Lecture

Terminology. Let $(X, \mathcal{B}, \mu)$ be a measure space and let P be a property of points in X. We say that

- a.e. point $x \in X$ satisfies P $\iff$ $\{x \in X : x \text{ satisfies } P\}$ is conull
  P holds a.s. (almost surely)
  P holds a.e. (almost everywhere)
- P holds for μ-positive set of points in X $\iff$ $\exists$ measurable set of points of positive measure for which P holds.

Prop (Ctbl pigeonhole). Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let $\mathcal{C}$ be an almost disjoint collection of sets of positive measure. (Almost disjoint means that any two distinct $A, B \in \mathcal{C}$ have null intersection.) Then $\mathcal{C}$ is ctbl.

Proof. First suppose μ is finite.

The Beatles "A day in my life": And though the holes were rather small,
The had to count them all.

Let $\mathcal{C}_n := \{A \in \mathcal{C} : \mu(A) \geq \frac{1}{n}\}$. Then each $\mathcal{C}_n$ is finite, in fact $|\mathcal{C}_n| \leq n \cdot \mu(X)$. But $\mathcal{C} = \bigcup_{n \geq 1} \mathcal{C}_n$, hence $\mathcal{C}$ is ctbl.

Now assume μ is σ-finite so $X = \bigsqcup_{n \in \mathbb{N}} X_n$, where $X_n \in \mathcal{B}$ and $\mu(X_n) < \infty$. Then $\tilde{\mathcal{C}}_n := \{A \in \mathcal{C} : \mu(A \cap X_n) > 0\}$ is ctbl by what we've proved already, and $\mathcal{C} = \bigcup_{n \in \mathbb{N}} \tilde{\mathcal{C}}_n$ since if $A \cap X_n$ is null for all n then A is null. Thus $\mathcal{C}$ is ctbl. $\square$

Corollary (Measure exhaustion). Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let $(A_\alpha)_{\alpha < \omega_1}$ be an increasing sequence of measurable sets.

Then this sequence stabilizes at a ctbl ordinal, i.e. $\exists$ ctbl ordinal $\beta < \omega_1$ s.t. $\forall \alpha > \beta$ $\quad A_\alpha =_\mu A_\beta$ (i.e. $A_\alpha \setminus A_\beta$ is null).

Proof. Disjointify and apply the ctbl pigeonhole: $\tilde{A}_\alpha := A_\alpha \setminus \bigcup_{\gamma < \alpha} A_\gamma$, then $\{\tilde{A}_\alpha : \alpha < \omega_1\}$ is a disjoint collection, so all but ctbly many $\tilde{A}_\alpha$ are null by the previous proposition. Let β be the sup of α with $\mu(\tilde{A}_\alpha) > 0$, so β is still a ctbl ordinal. $\square$

Remark. The last corollary allows using greedy algorithms to exhaust the space.

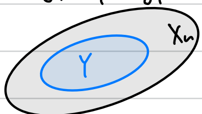
Def. Let $(X, \mathcal{B}, \mu)$ be a measure space. A set $A \in \mathcal{B}$ is called an atom if $\mu(A) > 0$ but $\nexists$ measurable subset $B \subseteq A$ with $0 < \mu(B) < \mu(A)$.
A measure space $(X, \mathcal{B}, \mu)$ is atomless if it has no atoms.

Sierpinski's Theorem. In an atomless measure space $(X, \mathcal{B}, \mu)$, every value $r \in [0, \mu(X)]$ is achieved by μ, i.e. for each $r \in [0, \mu(X)]$, there is $B \in \mathcal{B}$ with $\mu(B) = r$.

Proof. We first prove:

Claim 1. Every non-null $X' \in \mathcal{B}$ contains arbitrarily small sets of positive measure.

Proof. Because X isn't an atom, there is $X_0 \subseteq X'$ with $0 < \mu(X_0) < \mu(X') \leq \infty$.
Suppose X_n is defined, let $Y \subseteq X_n$ with $0 < \mu(Y) < \mu(X_n)$. Then either $\mu(Y) \leq \frac{1}{2}\mu(X_n)$ or $\mu(X_n \setminus Y) \leq \frac{1}{2}\mu(X_n)$, and let X_{n+1} be one of Y or $X_n \setminus Y$ so that $\mu(X_{n+1}) \leq \frac{1}{2}\mu(X_n)$. Thus we get $(X_n) \subseteq \mathcal{B}$ with $0 < \mu(X_n) \leq \frac{1}{2^n}\mu(X_0)^2$. $\square$ (Claim 1)



Let $r \in (0, \mu(X))$ and let $\mathcal{B}_r := \{B \in \mathcal{B} : 0 < \mu(B) \leq r\}$. Let

$$r' := \sup\{\mu(B) : B \in \mathcal{B}_r\}.$$

If this supremum r' is achieved by some $B \in \mathcal{B}_r$, i.e. $\mu(B) = r'$, then

$r' = r$ because otherwise we could take a set $A \subseteq X \setminus B$ with $0 < \mu(A) < r - r'$ and $A \sqcup B$ would have measure $r' < \mu(A \sqcup B) < r$, contradicting the def of r'. Thus, it remains to prove the following:

Claim 2. There is $B \in \mathcal{B}_r$ with $\mu(B) = r'$.

Proof 1. Suppose towards a contradiction there is no such set.

We do measure exhaustion: by transfinite induction define a sequence $(B_\alpha)_{\alpha < \omega_1}$ of pairwise disjoint sets so that $\bigcup_{\alpha < \beta} B_\alpha \in \mathcal{B}_r$ and $\mu(B_\alpha) > 0$. This yields a contradiction as in the proof of ctbl pigeonhole for a finite measure, while here we use that $\mu(\bigcup_{\alpha<\beta} B_\alpha) \leq r$ $\forall \beta < \omega_1$. We leave the details as an optional exercise. $\square$ (Proof 1)

Proof 2 ($\frac{1}{2}$-greedy measure exhaustion). We inductively define a sequence $(B_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets of positive measure with $\mu(\bigsqcup_{n<k} B_n) \leq r'$ for all $k \in \mathbb{N}$ as follows: take $B_0 \in \mathcal{B}_r$ hence $0 < \mu(B_0) \leq r'$.

Suppose $(B_i)_{i<n}$ is defined. We take $B_n \subseteq X \setminus \bigsqcup_{i<n} B_i$ so that $0 < \mu(B_n) \leq r' - \mu(\bigsqcup_{i<n} B_i)$ but B_n is $\geq \frac{1}{2}$ of the available measure, i.e.

$$\mu(B_n) \geq \frac{1}{2} \sup \left\{ B \in \mathcal{B} : 0 < \mu(B) < r' - \mu\left(\bigsqcup_{i<n} B_i\right) \right\}. \qquad (*)$$

We check that $B_\infty := \bigsqcup_{n \in \mathbb{N}} B_n$ then $\mu(B_\infty) = r'$. Indeed, firstly, $\mu(B_\infty) = \lim_{k \to \infty} \mu(\bigsqcup_{n<k} B_n)$ by monotone convergence, so $\mu(B_\infty) \leq r'$.

If $\mu(B_\infty) < r'$, then $\exists B' \subseteq X \setminus B_\infty$ with $0 < \mu(B') < r' - \mu(B_\infty)$. But note that $\sum_n \mu(B_n) = \mu(B_\infty) \leq r'$ hence $\lim_{n \to \infty} \mu(B_n) = 0$, so there is $n \in \mathbb{N}$ with $\mu(B_n) < \frac{1}{2}\mu(B')$, contradicting our $\frac{1}{2}$-greedy choice of the set B_n. $\square$ (Claim 2)

$\square$ (Theorem)

Regularity and tightness of measures.

Let X be a metric space and let μ be a Borel measure on X, i.e. $\mathcal{B} := \mathcal{B}(X)$.

Def. μ is said to be

(i) regular if for every measurable set B,

$$\mu(B) = \inf\{\mu(U) : U \supseteq B \text{ open}\} \quad \text{(outer regular)}$$
$$= \sup\{\mu(C) : C \subseteq B \text{ closed}\} \quad \text{(inner regular)}$$

(ii) strongly regular if for every measurable set B,

$$0 = \inf\{\mu(U \setminus B) : U \supseteq B \text{ open}\}$$
$$= \inf\{\mu(B \setminus C) : C \subseteq B \text{ closed}\}.$$

Observation. For finite measure spaces, regularity and strong regularity are equivalent.

By definition, we know that each measurable set A is contained and contains Borel sets B_1 and B_0 with $B_0 =_\mu A =_\mu B_1$. Strong regularity gives something stronger:

Observation. If μ is strongly regular, then for each measurable set A, there are a G_δ set G and F_σ-set F with $F \subseteq A \subseteq G$ so that $F =_\mu A =_\mu G$.

Proof. Let $U_n \supseteq A$ be open with $\mu(U_n \setminus A) < \frac{1}{n}$, so $G := \bigcap U_n$ is G_δ and $\mu(\bigcap_{n \in \mathbb{N}} U_n \setminus A) \leq \mu(U_m \setminus A) < \frac{1}{m} \to 0$ as $m \to \infty$. Similarly with closed sets. $\square$